INFLUENCE OF THE SURFACE SHAPE AND INDENTOR SIZE ON THE CRITICAL CONDITIONS OF BRITTLE FRACTURE

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1. At this time the question of the contact of paraboloids (the Hertz problem) [1] has been experimentally and theoretically investigated. The specific distribution of the stresses near the area of contact in detail permits the estimation of the fracturing load in this problem by both the criteria of the beginning of plastic flow [2] and for brittle fracture [3].

The influence of the shapes of compressible bodies on the magnitude of the maximum pressure achievable in the contact plane prior to the beginning of plastic flow was considered in [4] in an example of axisymmetric bodies bounded by the surface

$$z = Ar^{2\lambda}; \tag{1.1}$$

the z axis is perpendicular to the contact plane, r is the radius vector in the contact plane. For $\lambda = 1$ such a contact problem goes over into the Hertz problem. The results of this research show that the pressure in the contact plane can be raised to the limit value p_* achievable by the Mises criterion [5] if the maximum shear stress intensity is reached on the abutting surface and agrees with the site of the action of the greatest normal contact pressure. The possibility of realizing such a state of stress is independent of the value of the dimensional coefficient A governing the scale of the surface (1.1). A necessary condition for reaching the pressure p_* is the selection of the exponent λ in (1.1) from the range $1/2 < \lambda \leq \lambda^*$ ($\lambda^* < 1$ is indeed determined by the strength and elastic characteristics of the material).

The developed representation of the nature of microcrack propagation during brittle fracture under a spherical indentor [3] is extended in this paper to the case of an indentor whose surface is described by the power-law function (1.1). The results obtained are applied to estimate the pressure in a miniature chamber [6] where high pressure is produced during impressing a smooth conical indentor into a slab. Such a modification of the extensively utilized apparatus of the "Bridgman anvil" type is promising for the solution of problems about expanding the range of pressures accessible to research. Transition of the dielectrics BN, C, SiO_2, and MgO into the metallic state, as observed on this apparatus, is reported in [7].

2. The brittle fracture criterion can be obtained from the energy balance equation [5]. In this case, the change in free energy F associated with the presence of a crack of length c is investigated. The crack becomes unstable if its dimension $c = c_k$ corresponds to one of the extremums of the function F(c). As the tensile force grows, the crack grows quasistatically if c_k determines the minimum F(c); the critical crack is lengthened spontaneously under invariant external conditions if c_k corresponds to the maximum F(c).

In such an approach to the problem, two parameters of the dimensionality of a length are introduced from the beginning: the mean dimension c of cracks existing in the material, and the ratio γ/E, whose value is determined by the interatomic binding force. The nature of the development of a dangerous crack depends on the relationship between c, γ/E, and the dimension of the domain of elevated tensile stresses in which this crack is located.

During contact the values of c and γ/E should be commensurate with the dimension of the contact spot a_T during fracture. For indentors bounded by the surface (1.1), the value of a_T for a given value of λ is determined by the magnitude of the coefficient A characterizing the size of the indentor. As will be shown below, the "dimension" of the bodies making contact substantially affects the magnitude of the pressure attainable in the contact plane up to the time of brittle fracture.

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Near the center of contact all the principal values of the stress tensor correspond to compression, which hinders crack development. Brittle fracture usually starts on the contact surface outside the area of contiguity ($z = 0$, $r \geq a$, a is the radius of contact), where pure shear is realized under plane stress state conditions: The components σ_{zz} and σ_{rz} of the stress tensor vanish, but $\sigma_{\varphi\varphi} = -\sigma_{rr}$. The most probable site of the beginning of brittle fracture is the edge of the contact spot where the tensile stress σ_{rr} reaches its maximum value σ_m:

$$\sigma_m = \frac{1-2\nu}{2}\frac{Q}{\pi a^2} \tag{2.1}$$

(Q is the magnitude of the compressive force and ν is Poisson's ratio). The stress σ_m rises with the growth of Q. The formation of circular conical cracks along the edge of the contact spot is observed in experiment.

The stress field near the contact surface is conveniently described by the principal values of the tensor σ_{ik}. Their numbering is chosen so that the principal stresses σ_1 and σ_2 agree on the surface $z = 0$ with the stresses σ_{rr} and σ_{zz}, respectively. The stress $\sigma_{\varphi\varphi}$ agrees with the principal value σ_2.

It is shown [8] in experiments with spherical indentors that in a first approximation a conical crack develops normally to the principal tensile stress σ_1. It is assumed below that this regularity is satisfied even for cracks being formed under an indentor whose surface is described by (1.1).

For arbitrary axisymmetric indentors, the contact pressure around the edge of the area of contiguity ($r \to a$) decreases as $(1 - r/a)^{1/2}$. This general dependence should even be felt by the rate of stress decrease near the contact edge. For verification by the known Boussinesq formulas [1], the stresses under an indentor bounded by the surface (1.1) were computed numerically for different values of λ, and the dependences $\sigma_1 = \sigma_1(l)$ governing the tensile stress σ_1 distribution along the direction of crack development assumed (Fig. 1). The most convenient representation for the normal pressure distribution under an indentor of such shape is obtained in [9]. Following [3], where the formation of a conical crack during impression of a spherical indentor into a slab was considered, we write $\sigma_1(l)$ in the form

$$\sigma_1(l) = \begin{cases} \sigma' + (\sigma_m - \sigma')(1 - l/\delta), & l < \delta, \\ \sigma', & l > \delta, \end{cases} \tag{2.2}$$

which adequately describes the value of σ_1 for $l \leqslant 0.5a$; the spacing l is measured from the contact surface. The ratios δ/a and σ'/σ_m depend slightly on λ: $\delta/a \approx 0.13$, $\sigma'/\sigma_m \approx 0.1$.

The influence of the indentor shape (the value of λ in (1.1)) on the critical brittle fracture parameter is determined by the dependence of the size of the contact spot, and therefore, by the magnitude of the stress σ_m (2.1) on the indentor geometry for a given load Q. Below, the results of [3] are written in a form accessible to extension to the case of fracture under an axisymmetric indentor of arbitrary shape.

Solutions of the equation $\partial F(c)/\partial c = 0$ in the stress field (2.2) are investigated on the basis of the theory of fracture [10]. The function $F(c)$ has three extremums for $c = c_k$; the values of c_k are c_0, c_1, c_2 if σ_m does not exceed the critical value σ_m^*:

$$\sigma_m^* \approx \left[\frac{1}{2}\left(\frac{\sigma_m}{\sigma'}\right)\left(\frac{a}{\delta}\right)\frac{1}{1-\nu^2}\right]^{1/2}\sqrt{\frac{\gamma E}{a}} \sim 6\sqrt{\frac{\gamma E}{a}}. \tag{2.3}$$

The dimension

$$c_0 \approx c_G\left[1 + \frac{4}{\pi}\frac{c_G}{\delta}\right], \quad c_G = 2\gamma E/\pi(1-\nu^2)\sigma_m^2, \tag{2.4}$$

is determined by the maximum of $F(c)$ for $c_0 \ll \delta$. The value c_G agrees with the critical size of the Griffith crack [5] under a constant tensile stress $\sigma = \sigma_m$. The values of $c_k = c_{1,2} > \delta$ indeed determine the minimum and maximum of the function $F(c)$, respectively. For $\sigma_m = \sigma_m^*$ these extremums merge: $c_1 = c_2 = c^*$; as the load grows the equation $\partial F(c)/\partial c = 0$ has no real roots for $c > \delta$:

$$c^* \approx \frac{1}{4}\left(\frac{\sigma_m}{\sigma'}\right)^2 c_G^*. \tag{2.5}$$

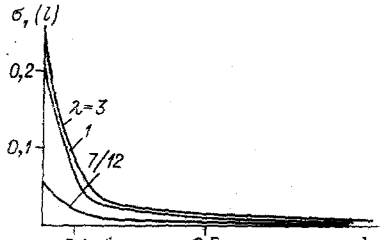

Fig. 1

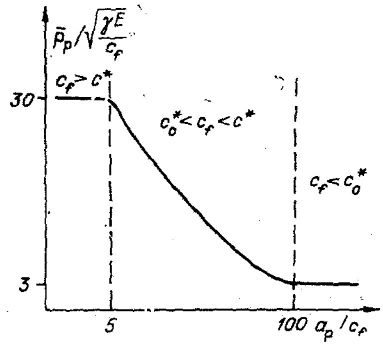

Fig. 2

The asterisk marks values of all the parameters for $\sigma_m = \sigma_m^*$. The change in the number of possible equilibrium sizes of carck for $\sigma_m = \sigma_m^*$ is related to the change in the stability conditions for surface cracks, for which a spontaneous increase in whose dimensions will result in the appearance of a developed conical crack.

Let c_f be the mean size of a surface crack in the undeformed material. The difference in the nature of the propagation of such cracks for different relationships between c_f and the characteristic dimensions c_0^*, c^* can be illustrated most clearly by a graph of the dependence of the mean pressure $\bar{p}_r = Q_r/\pi a_r^2$ on the size of the contact spot a_r up to the time of fracture. Such a dependence is universal and should be observed for arbitrary axisymmetric indentors. For a given value of σ_m the mean pressure is determined by means of (2.1).

1) Let $c_f < c_0^*$. In this case, crack growth starts for some value $\sigma_m = \sigma_m^1 > \sigma_m^*$, when the critical dimension c_0 diminishes to the value c_f. A surface crack, which becomes unstable for $\sigma_m > \sigma_m^*$, lengthens, spontaneously causing complete fracture. It follows from (2.4) and (2.5) that $\sigma_m^1 = \sqrt{2\gamma E/[\pi(1-\nu^2)c_f]}$. The condition $c_f < c_0^*$ governs the possible values of the radius of the contact spot during fracture: $a_r > a_1 = (\pi/4)(\sigma_m/\sigma')(a/\delta)c_f \sim 100c_f$ (Fig. 2).

2) If $c_0^* < c_f < c^*$ (or $a_2 < a_r < a_1$; $a_2 = \pi(\sigma'/\sigma_m)(a/\delta)c_f \sim 5c_f$), the conical crack will develop in two stages. First, when the value of c_0 becomes equal to c_f (for $\sigma_m^1 < \sigma_m^*$), the surface crack will grow to a size corresponding to the equilibrium value c_1. This increase in the crack depth is insignificant and not perceived as complete fracture. Under further loading, the crack dimension increases in equilibrium until it agrees with the value c_2 ($c_1 = c_2 = c^*$) for $\sigma_m = \sigma_m^*$. Here the crack size grows abruptly, resulting in fracture. The maximum tensile stress to the time of fracture is σ_m^* (Fig. 2).

3) If the radius of the contact spot during fracture is sufficiently small ($a_r < a_2$), which corresponds to the inequality $c_f > c^*$, the domain of elevated stresses δ near the surface is too small, and the development of the surface crack occurs in a practically constant stress field σ'. Fracture starts for $\sigma_m < \sigma_m^*$ when the value of the unstable dimension c_2 is commensurate with c_f (Fig. 2).

To determine the magnitude of the fracturing load Q_r as a function of the indentor geometry, the relationships (2.3)-(2.5) should be supplemented by an equation governing the radius of the contact spot as a function of the load. If the indentor surface is described by (1.1), then in conformity with [9]

$$a^{2\lambda+1} = \frac{Q(1-\nu^2)\,\Gamma(\lambda+3/2)}{\sqrt{\pi}\,\Gamma(\lambda)\,\lambda^2 AE}. \tag{2.6}$$

The values of c_0^* and c^* computed by means of (2.4) and (2.5) are constants characterizing this indentor. Inequalities bounding possible values of the dimensional coefficient A governing the indentor size can be compared by different relationships between c_f, c_0^* and c^*. Using (2.6), we obtain

$$\begin{cases} c_f < c_0^*(\lambda), & \text{if} \quad A < A_0^{(\lambda)} \equiv \frac{2^{4\lambda-1/2}}{1-2\nu}\frac{\Gamma(\lambda+3/2)}{\lambda^2\Gamma(\lambda)}\left[\pi\left(\frac{\sigma_m}{\sigma'}\right)\left(\frac{a}{\delta}\right)\right]^{1-2\lambda}\sqrt{\frac{\gamma(1-\nu^2)}{Ec_f^{4\lambda-1}}}, \\ c_0^*(\lambda) < c_f < c^*(\lambda), & \text{if} \quad A_0^{(\lambda)} < A < A_0^{(\lambda)}\left(\frac{\sigma_m}{2\sigma'}\right)^{4\lambda-1}, \\ c_f > c^*(\lambda), & \text{if} \quad A > A_0^{(\lambda)}\left(\frac{\sigma_m}{2\sigma'}\right)^{4\lambda-1}. \end{cases} \tag{2.7}$$

Summarizing the previous results, we write an expression for the fracturing load $Q_r^{(\lambda)}$ as a function of the parameters governing the indentor shape

$$Q_r^{(\lambda)} = \begin{cases} Q_\lambda^*\left[\frac{c_0^*(\lambda)}{c_f}\right]^{\frac{2\lambda+1}{2(2\lambda-1)}}, & A < A_0^{(\lambda)}, \\ Q_\lambda^*, \; A_0^{(\lambda)} < A < A_0^{(\lambda)}\left(\frac{\sigma_m}{2\sigma'}\right)^{4\lambda-1}, & \\ Q_\lambda^*\left[\frac{4c^*(\lambda)}{c_f}\right]^{\frac{2\lambda+1}{2(2\lambda-1)}}, & A \gg A_0^{(\lambda)}\left(\frac{\sigma_m}{2\sigma'}\right)^{4\lambda-1}, \end{cases} \tag{2.8}$$

$$Q_\lambda^* = \left[\frac{E^{2(\lambda-1)}\gamma^{2\lambda+1}}{A^3}\left(\frac{(1-\nu^2)\Gamma(\lambda+3/2)}{\lambda^2\Gamma(\lambda)\sqrt{\pi}}\right)^3\left(\frac{2\pi^2\left(\frac{\sigma_m}{\sigma'}\right)\left(\frac{a}{\delta}\right)}{(1-\nu^2)(1-2\nu)^2}\right)^{2\lambda+1}\right]^{\frac{1}{4\lambda-1}}$$

The explicit form of $Q_r^{(\lambda)}$ for $A > A_0^{(\lambda)}(\sigma_m/2\sigma')^{4\lambda-1}$ is obtained successfully only asymptotically for $A \gg A_0^{(\lambda)}(\sigma_m/2\sigma')^{4\lambda-1}$, when $c_f \gg c^*(\lambda)$. In the particular case $\lambda = 1$ (a spherical indentor of radius $R = 1/2A$), the estimates obtained above for the critical fracture conditions agree with the results in [3].

3. The expression (2.8) for the limit load $Q_r^{(\lambda)}$ permits estimation of the maximum pressure being developed in the contact plane up to the time of brittle fracture. Taking account of the results in [4], such computations can be used as the initial data for the construction of high-pressure apparatus of the "Bridgman anvil" type.

Comparing the fracture conditions under indentors whose shape is described by the equation $z = Ar^{2\lambda}$ for different λ, it can be seen that fracture is made extremely difficult if the indentor surface is almost conical ($\lambda \to 1/2$): The effects of plastic flow here appear only for high limit pressures (see Sec. 1); for a given normal pressure $P(0)$ at the center of contact, the value of the maximum tensile stress σ_m governing the appearance of brittle cracks on the edge of the contact spot tends to zero as $\lambda \to 1/2$: $\sigma_m(\lambda) = (1-2\nu) \times [(2\lambda-1)/(2\lambda+1)]P(0)$.

It can be shown that for $\lambda \le 1$ the maximum normal pressure $P_\lambda^m = [(2\lambda+1)/2(2\lambda-1)](Q/\pi a^2)$ is reached at the center of contact. The value of P_λ^m is determined by the known mean pressure $\bar{P}_r$ up to the beginning of brittle fracture (see Fig. 2). For a given λ the increase in the contact pressure can be expected in only a small contact spot ($a_r < a_1 \sim 100c_f$), which corresponds to the values $A > A_0^\lambda$ (see (27)).†

For values of λ close to 1/2, the pressure distribution in the contact plane is acutely inhomogeneous, which permits obtaining a pressure near the center which is many times greater than the mean loading over the area. The rise in the maximum pressure in the contact plane is associated with diminution of the size of the domain in which the pressure is high.

A rounded-off conical indentor

$$z(r) = \begin{cases} \frac{r^2}{2R}, & r \le r_0, \\ r\,\mathrm{tg}\,\beta + \left(\frac{r_0^2}{2R} - r_0\,\mathrm{tg}\,\beta\right), & r \ge r_0 \end{cases} \tag{3.1}$$

†For spherical indentors ($\lambda = 1$, $A = 1/2R$), the increase in the coefficient A corresponds to diminution in the indentor radius.

(the smooth part of the indentor goes smoothly over into a conical part at the point r_0: $r_0 = R \tan \beta$, $\beta = \pi/2 - \alpha$, 2α is the cone aperture angle) and a slab of artificial diamonds of the carbonado type in the apparatus of [6] were used as working elements of a high-pressure chamber.

Possibilities of the rounded-off cone-plane scheme have been discussed extensively in the literature. This question has been examined in greatest detail in [11], where the magnitude of the maximum pressure P_r^m being realized in such a chamber prior to the beginning of fracture is estimated by means of the value of the tensile stress at the edge of the contact spot within the framework of the Hertz theory, from which $P_r^m \sim 60$ kbar. Such an estimate is valid only for definite relationships between the round-off radius of the cone R and the aperture angle 2α, when R is sufficiently large and the fracturing load is less than the force needed for the conical indentor surface to take part in the contact. If the parameters of the system (3.1) allow participation of the conical surface in the contact prior to fracture, then the results of [11] are inapplicable; the value of P_r^m cited there is substantially lowered.

The normal pressure distribution P(r) under the rounded-off conical indentor can be determined by the general formula [12] relating the value of P(r) to the shape of an axisymmetric indentor by the integral relation

$$P(r) = -\frac{1}{2\pi}\int_r^a \frac{F'(s)\,ds}{\sqrt{s^2 - r^2}}, \qquad 0 < r \leqslant a, \tag{3.2}$$

$$F(s) = \frac{E}{1-\nu^2}\left[h - s\int_0^s \frac{z'(\sigma)\,d\sigma}{\sqrt{s^2 - \sigma^2}}\right]$$

(h is the indentor displacement). The radius of contact a is determined from the equilibrium condition

$Q = 2\pi \int_0^a P(r)\,r\,dr$, which can be reduced to the form

$$\int_0^a z'(\sigma)\,\sigma^2 (a^2 - \sigma^2)^{-1/2} d\sigma = Q(1 - \nu^2)/E.$$

An investigation of the expression (3.2), where z(r) is given by (3.1), shows that the conical indentor surface takes part in the contact only for loads $Q > Q_0$, where Q_0 is the force for which the radius of the contact spot is $a = r_0$:

$$Q_0 = \frac{2}{3}\frac{E \operatorname{tg} \beta r_0^2}{(1 - \nu^2)}.$$

For $Q < Q_0$ the pressure distribution in the plane of contact can be determined by means of the Hertz solution: $P(r) = (3/2)(Q/\pi a^2)\sqrt{1 - r^2/a^2}$. For loads $Q \gg Q_0$

$$P(r) = \frac{E \operatorname{tg} \beta}{4(1-\nu^2)}\begin{cases} \ln\left[\dfrac{1 + \sqrt{1 - r^2/a^2}}{r_0/a + \sqrt{r_0^2/a^2 - r^2/a^2}}\right], & r < r_0, \\ \ln\left(\dfrac{a}{r} + \sqrt{\dfrac{a^2}{r^2} - 1}\right), & r > r_0; \end{cases} \tag{3.3}$$

here $a \approx 2((1 - \nu^2)Q/\pi E \tan \beta)^{1/2}$.

For $r > r_0$ formula (3.3) describes the dependence of the contact pressure on the radius r for conical indentors ($r_0 = 0$). The logarithmic growth of the pressure at the center of contact can hence be obtained directly for $r_0 \ll a$ as a result of taking the average over a small area of radius r_0.

Polycrystalline diamonds of the carbonado type are a new ultrahard material for which brittle fracture is characteristic. The load at which fracture of the indentors fabricated from this material occurs can be estimated by means of (2.8).

The condition $Q_0 < Q_r^1$ in the system (3.1), where Q_r^1 is the fracturing load of a spherical indentor, governs the relationship between the parameters R and β, allowing participation of the conical indentor surface in the contact. The inequality $Q_0 < Q_r^1$ is satisfied if

$$\operatorname{tg}\beta < \frac{3}{1-2\nu}\sqrt{\frac{2\pi(1-\nu^2)\gamma}{Ec_f}}, \quad R > R_0,$$

$$\operatorname{tg}^3\beta R < \frac{9\pi^2}{2}\frac{\left(\frac{a}{\delta}\right)\left(\frac{\sigma_m}{\sigma'}\right)(1-\nu^2)(\gamma/E)}{(1-2\nu)^2}, \quad R < R_0, \tag{3.4}$$

where

$$R_0 = \frac{1-2\nu}{24}\left(\frac{a}{\delta}\right)\left(\frac{\sigma_m}{\sigma'}\right)\sqrt{\frac{2\pi Ec_f^3}{(1-\nu^2)\gamma}}.$$

The lack of confident data about the value of the surface energy γ and the mean size of the surface cracks c_f in diamonds of the carbonado type makes difficult a realistic estimate of the magnitude of the pressure in the scheme under consideration. However, if it is assumed that the value of c_f agrees in order of magnitude with the usual sizes of surface cracks in brittle materials [3] ($c_f \sim 10^{-4}$ cm) then the value of γ can be evaluated by using the data about the fracture of large-radius spherical indentors when it is sufficient to know the value of the tensile stress on the edge of the contact spot to estimate fracture (see Sec. 2). It is indicated in [6] that the mean pressure during fracture of balls of radius R ~ 2-2.5 cm is on the order of $4 \cdot 10^4$ kg/cm². For $E \sim 10^7$ kg/cm² (the value of Young's modulus for diamonds), this corresponds to the value $\gamma \sim 1.5 \cdot 10^{-3}$ kg/cm.

The results of computations [11] are valid only outside the domain (3.4) for $R > R_0$, $R_0 \sim 0.4$ cm. Upon compliance with the inequalities (3.4), the pressure P_r^m should be estimated by (3.3), where $Q = Q_r^{(1/2)}$; the value of $Q_r^{(1/2)}$ is determined from expression (2.8) for $\lambda = 1/2$. Here the pressure at the center of contact can reach values on the order of 10^6 kg/cm² without fracture. Thus, for instance, for a cone aperture angle $2\alpha \sim 160°$ and radius $R \sim 10^{-4}$ cm the pressure will be $P_r^m \sim 1.5 \cdot 10^6$ kg/cm².

It should be noted that the examination performed above of the fracture conditions assumes the presence of a surface crack of given size c_f at the edge of the contact spot for any size of the area of contiguity. This assumption is valid if $a \gg c_f$. If fracture occurs at the contact spot whose size is on the order of the spacing between surface cracks, then it can turn out that a crack of the size c_f will not be in the domain of elevated tensile stresses for $a = a_r$, and the indentor will sustain the load $Q > Q_r$. In this sense the values obtained for P_r^m must be understood as the lower bound for small contact spot sizes.

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COMPUTATION OF EFFECTIVE PLASTICITY CHARACTERISTICS OF INHOMOGENEOUS MEDIA

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Macroscopic mechanical characteristics of a composite material, representing a mixture of inclusions and host, are determined by the mechanical properties of the phases and its geometric configuration. We define the composite configuration by that uniform distribution of the spherical inclusions in the host so that the characteristic function $\varkappa$ equals one at the inclusions and zero at the host and is statistically homogeneous and isotropic. With respect to the mechanical properties of the phases, we limit ourselves to the condition that the plastic properties of the inclusions be higher than th; plastic properties of the host. Hence, the host can be considered ideally elastic in a definite deformation range, and the inclusions ideally elastic–plastic Both phases are interconnected such that slip of the inclusions in the host is excluded.

1. The materials of the host and the inclusions are considered isotropic and Hooke's law in the phases is written in the form

$$\sigma_{ij}=2\mu_\alpha\left(e_{ij}-e_{ij}^p\right)+\delta_{ij}\lambda_\alpha e_{kk},$$

where μ_α, λ_α are the Lamé parameters, σ_{ij}, e_{ij}, e_{ij}^p are components of the stress, the total and plastic deformation tensors, and $\alpha = 1$ corresponds to the host and $\alpha = 2$ to the inclusion. The plastic deformations satisfy the incompressibility condition $e_{kk}^p = 0$. The plastic properties of the inclusions are determined by the Mises plasticity condition $s_{ij}s_{ij} = k^2$, where s_{ij} and k are the deviator components of the stress tensor and the plasticity limit of the inclusions, respectively.

An investigation of the extremum [1] of the function

$$L=\frac{1}{V}\left\{\int\limits_V\left[D\left(\varepsilon_{ij}^p\right)+\frac{1}{2}W\left(e_{ij}-e_{ij}^p,\ \varepsilon_{ij}-\varepsilon_{ij}^p\right)\right]dV-\int\limits_S\left(p_i v_i+q_i u_i\right)dS\right\} \tag{1.1}$$

determines the properties of the inhomogeneous medium. Here $D(\varepsilon_{ij}^p) = k(x)\sqrt{\varepsilon_{ij}^p\varepsilon_{ij}^p}$ is the dissipative function for the selected plasticity condition [2];

$$\frac{1}{2}W\left(e_{ij}-e_{ij}^p;\ \varepsilon_{ij}-\varepsilon_{ij}^p\right)=2\mu(x)\left(e_{ij}-e_{ij}^p\right)\left(\varepsilon_{ij}-\varepsilon_{ij}^p\right)+\lambda(x)e_{kk}\varepsilon_{kk}$$

is the rate of change of the elastic energy, ε_{ij}, ε_{ij}^p are the components of the total and plastic strain rate tensors, u_i, v_i are the displacements and the velocities, p_i, q_i are the loads and their velocities on the surface. The total volume V is a simply connected domain. The random stress, strain, and their velocity fields

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